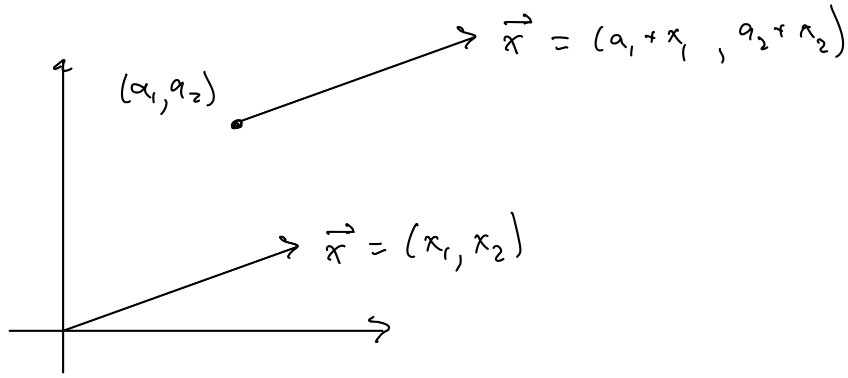
2/2/26

<u>Appendix A</u> ~ some highlights

we've talked about lin. combinations of vectors and I asked you to look at theorem A.2 so we'll assume that's OK. Briefly talk about the rest.

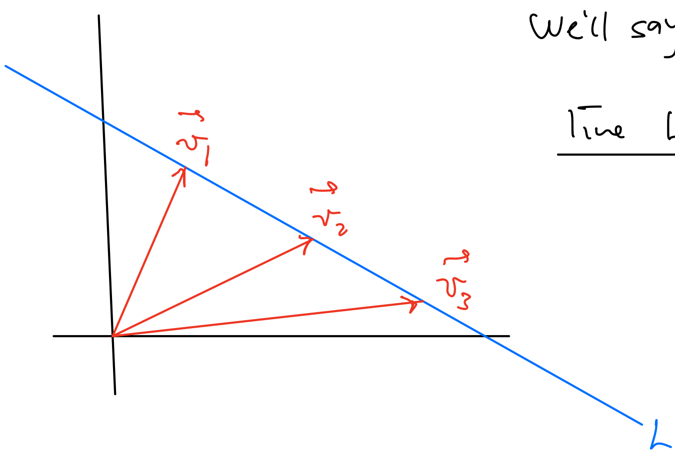
we've talked about viewing linear combinations of vectors geometrically in $\mathbb{R}^2$. this is based on the <u>standard representation</u> of vectors, thinking of them as starting at the origin.

We can think of a <u>translation</u> of such a vector:



For a standard rep'n we usually won't distinguish between the vector $\vec{x}$ and its endpoint (x_1, x_2) (when the starting point is the origin)

<u>Ex</u>



We'll say $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ in standard rep'n <u>lie on the line L</u> because their endpoints do.

Strange def: 2 vectors $\vec{v}$, $\vec{w}$ are <u>parallel</u> if one is a scalar mult of the other (in $\mathbb{R}^n$)

Ex $5\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}5\\10\\15\\20\end{bmatrix}$ so $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ is parallel to $\begin{bmatrix}5\\10\\15\\20\end{bmatrix}$

$0\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ so $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ and $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ are parallel.

In fact $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ is parallel to anything in $\mathbb{R}^4$!

We talked about the dot product (when it's defined, it's always a scalar):

If the components of $\vec{v}$ are $v_1, \dots, v_n$

and " " " $\vec{w}$ are $w_1, \dots, w_n$ in $\mathbb{R}^n$ then

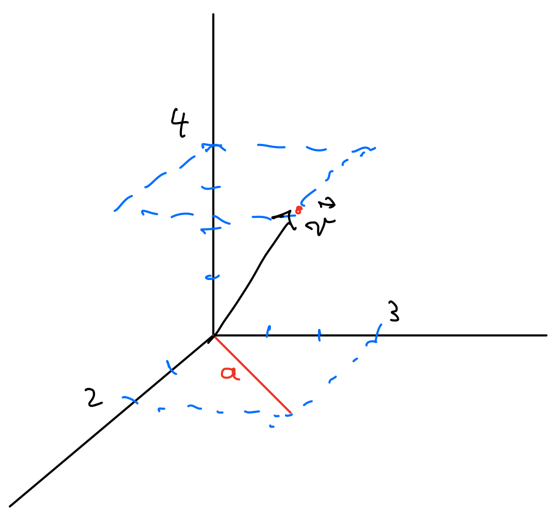
$$\vec{v} \cdot \vec{w} = v_1 w_1 + \dots + v_n w_n$$

this has another interpretation

Note $\|\vec{v}\| = \sqrt{v_1^2 + \dots + v_n^2}$ - this is the <u>length</u> or <u>norm</u> of $\vec{v}$

For example:

$\vec{v} = \begin{bmatrix}2\\3\\4\end{bmatrix} \in \mathbb{R}^3$



$a^2 = 2^2 + 3^2$

$\|\vec{v}\|^2 = a^2 + 4^2$

$= 2^2 + 3^2 + 4^2$

then
$$\boxed{\vec{v}\cdot\vec{w} = (\|\vec{v}\|)(\|\vec{w}\|)\cos(\theta)}$$

where θ is the angle between $\vec{v}$ and $\vec{w}$. (Details in Ch 5.)

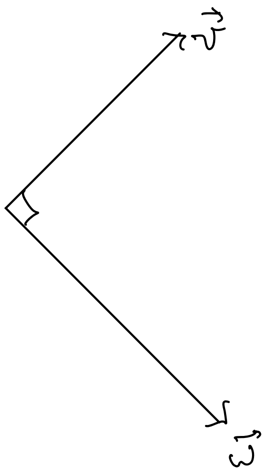
$\boxed{\text{Th. A.5 gives basic facts about dot prod.}}$ be sure to look at it

Note - in $\mathbb{R}^n$, if $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ then $\vec{x}\cdot\vec{x} = x_1^2 + \cdots + x_n^2 = \|\vec{x}\|^2$

In partic, the angle θ between $\vec{x}$ and $\vec{x}$ is 0 and $\cos(0) = 1$,

so the formula gives again: $\vec{x}\cdot\vec{x} = (\|\vec{x}\|)(\|\vec{x}\|)\cos(\theta) = \|\vec{x}\|^2$

Another special case: if $\vec{v}$, $\vec{w}$ are perpendicular



$$\vec{v}\cdot\vec{w} = (\|\vec{v}\|)(\|\vec{w}\|)\cos(90^\circ) = 0$$

(The vectors don't have to be in $\mathbb{R}^2$ for this to be true.)

In $\mathbb{R}^3$ we have a different kind of product — the cross-product.

while the dot product is a scalar, the cross-product is again a vector.

(But it's restricted to $\mathbb{R}^3$.)

You can read about it in Appendix A but for now we'll skip it.

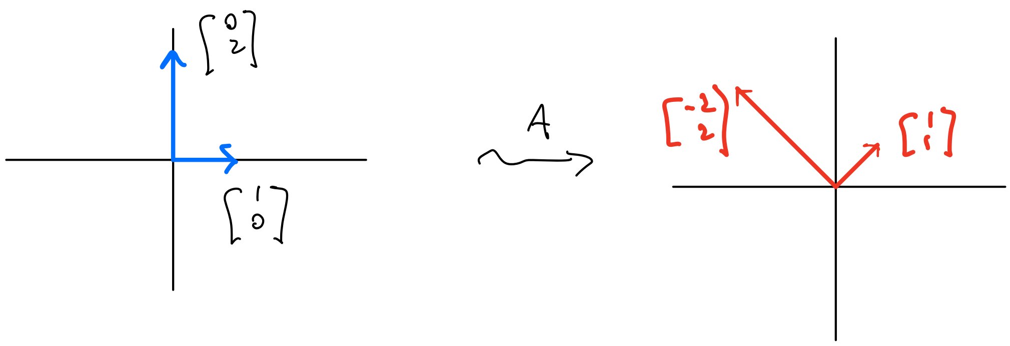
Now focus on §2.2. Special lin. transfs coming from geometry

More examples of lin transf and their geometric effects can be foun at the beginning of §2.2, studying the "standard L" configuration. Here's one.

Ex $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$



geometric effect: rotate by $-45°$ and a stretch ("scaling")

1st Special kind of linear transformation: orthogonal projection.

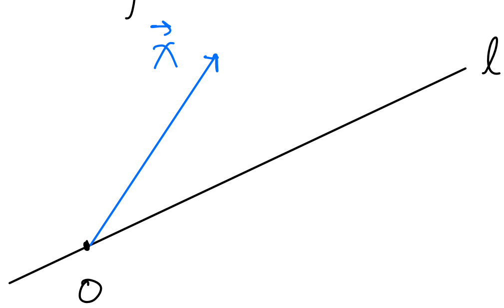
1st describe it geometrically, then see how the algebra works.

Several steps.

Big picture:

let ℓ be a line in the plane thru the origin.

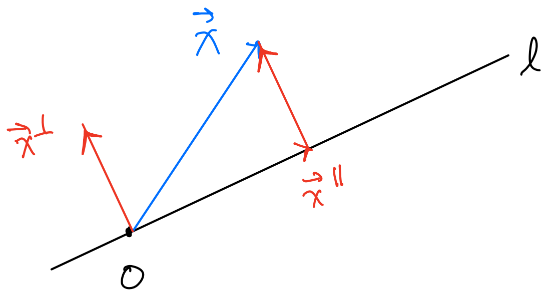
let $\vec{x}$ be any vector in $\mathbb{R}^2$



<u>Problem</u>: How can we write $\vec{x}$ as a sum of vectors

$$\boxed{\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}} \qquad (*)$$

where $\vec{x}^{\parallel}$ is parallel to ℓ (specifically it lies on ℓ) and $\vec{x}^{\perp}$ is perpendicular to ℓ?



Geometrically it's clear what we have to do, but how do we achieve it algebraically??

<u>Def</u> the transformation taking $\vec{x}$ to $\vec{x}^{\parallel}$ $(\mathbb{R}^2 \to \mathbb{R}^2)$ is called the <u>orthogonal projection</u> of $\vec{x}$ to ℓ. Sometimes we write

$$\vec{x}^{\parallel} = \mathrm{proj}_{\ell}(\vec{x}).$$

So how do we find $\vec{x}^{\parallel}$ or $\vec{x}^{\perp}$?

The idea is to find $\vec{x}^{\parallel}$ first. Then $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$ from $(*)$

So how do we find the orthogonal projection?

<u>Step 1</u> Choose any point W on ℓ. Let $\vec{w}$ be the vector defined by W.

e.g. ℓ is defined by $2x + 3y = 0$
b/c ℓ passes thru the origin

$W = (3, 2)$

$\vec{w} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

We know $\vec{x}^{\parallel}$ is parallel to ℓ, so $\vec{x}^{\parallel} = k\vec{w}$ for a suitable scalar k.
(we'll have to figure out how to find k.)

Step 2 We know $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel} = \vec{x} - k\vec{w}$. Use this to find k.

We know $\vec{x}^{\perp}$ is perpendicular to ℓ, so also to $\vec{w}$.

Then $(\vec{x} - k\vec{w}) \cdot \vec{w} = 0$

$\Rightarrow$ (by properties of the dot prod found in Th. A.5)

$$\vec{x} \cdot \vec{w} - k(\vec{w} \cdot \vec{w}) = 0$$

$$\Rightarrow \quad k(\vec{w} \cdot \vec{w}) = \vec{x} \cdot \vec{w}$$

$$\Rightarrow \quad k = \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}$$